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Role of time delay and harvesting in some predator–prey communities with different functional responses and intra-species competition

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ABSTRACT

We propose four predator–prey models: RM (Rosenzweig–MacArthur) model, BD model (RM type model with Beddington–DeAngelis functional response), RMI model (i.e., RM model with intraspecific competition among predators) and BDI model (BD model with intraspecific competition among predators). Each model incorporates time delay in the predators' numerical response. We first analyse the delay-induced stability for all the models. We show that increasing delay always destabilizes a coexisting stable equilibrium in RM and BD models. However, increasing delay does not always destabilize a stable equilibrium in RMI and BDI models. Indeed, the stable equilibrium, in the latter two models, may also maintain its stability due to varying delay. Thus, one of the major conclusions is that the invariance property of the local stability in RMI and BDI models is due to the influence of intraspecific competition. Analytically, we prove that stability switching is impossible to occur in all the models. Later, we implement harvesting of the prey and predator separately, which may generate stability switching. If populations oscillate in the unharvested system, extensive effort has a potential to stabilize the equilibrium. Under the same natural condition (unharvested situation), prey harvesting and predator harvesting may produce opposite dynamic modes.

1. Introduction

Investigating the dynamical relationship between prey and predator has been of utmost importance in population dynamics. It is the interaction between the predators and preys that dictates the nature of the system. To explore the dynamics of the predator and prey, several types of mathematical models have been proposed [1–4]. One of the crucial components of the predator–prey relationship is the predator's feeding behavior upon its prey. This fact is modeled by considering different functional and numerical responses. Although Holling type I and II functional responses are widely studied to explain many ecological results, Beddington–DeAngelis functional response [5,6] have become more popular in the last few decades as it reflects predators' interference during the predation process. The main reason behind considering Beddington–DeAngelis functional response for the predation is due to its generality. The Holling type II response can be derived as a particular case in the absence of mutual interference between predators. Predator–prey models with Beddington–DeAngelis functional response could display more complex dynamics in comparison to the models involving Holling type II response. During the

course of predation process, the time wasted upon encounters with other predators is the main source of deriving Beddington–DeAngelis functional response [5]. Time lags occur in many physical, mechanical and engineering disciplines including population dynamics models [7]. Therefore, to capture more realistic features of ecological systems, researchers incorporated the time delay factor in mathematical models. Mathematical models in ecology involves time delay in the growth rate of prey species [8–10], predator functional response [8,11], dispersal term [12–14], etc. These contributions established the mathematical conditions for the permanence of the systems, global stability, the periodicity of the solutions, existence of Hopf bifurcation, stability switching, etc.

Chen and You [15] concentrated on the permanence, extinction and periodic solution of the system with Beddington–DeAngelis functional response and stage structure for prey. It is shown by Liu [16] that increasing the delay always causes Hopf bifurcation in a modified Leslie–Gower predator–prey model incorporating Beddington–DeAngelis functional response. Li et al. [17] developed predator–prey dynamics with stage structure for prey populations where time delay plays

a role in the transition of immature prey to mature class. A geometric method was applied to show the existence of stability switching in the system dynamics. Li et al. [9] found the existence of multiple stability delay intervals, i.e., stability switching phenomenon in a two species model. Dubey et al. [18] have established a global stability result in a predator–prey system under two discrete time delays. Conditions for local stability and global stability of the feasible equilibrium are derived by Maiti et al. [19] in a delayed stage structure predator–prey system involving Crowley–Martin-type functional response. A predator–prey model with Allee effect, intraspecies competition and double-delay has been studied by Banerjee et al. [20]. They identified the appearance of a subcritical Hopf bifurcation due to the time delay in the presence of combined influence of intraspecies competition in the predator and Allee effect. A delayed model of predator and prey (with Allee effect) has been investigated by Anacleto and Vidal [21] to show the existence of stability switching analytically and establish the direction of Hopf bifurcation with respect to time delay. They have found that the delay due to the gestation period can be effected by the Allee effect, leading to a stability switching around the coexisting equilibrium. Mondal and Samanta [22] have considered a predator–prey model with Crowley–Martin functional response with two time delays, viz., logistic delay and gestation delay. They claimed through numerical simulations that gestation delay cannot destabilize the system. Barman and Ghosh [14] formulated a two-patch predator–prey model by combining density-dependent and density-indepndent dispersal. It was shown that increasing time delay in dispersal could stabilize (destabilize, respectively) an unstable (stable, respectively) coexisting equilibrium and even induce switching of stability.

Relatively less attention has been paid in investigating the explicit effects of harvesting in models incorporating time delays. It is of importance to understand the dynamic mode of several models in the context of fishery and biological pest control theory prior to harvest and introduction of control tactics, respectively. Thus, we focus on addressing the possible dynamic mode of predator–prey systems under varying time delay and effort exerted on harvesting. Kar and Pahari [23] developed a delayed predator–prey model combined with Beddington–DeAngelis functional response. The predator was subject to harvesting. It was observed that time delay causes stability switching. They also found, by computing the numerical solutions of the model ODEs, that harvesting destroyed cyclic dynamics. A similar model was proposed by Qu and Wei [24] to present more analytical results. They successfully

proved that appropriate choice of effort prevented the cyclic dynamics of the system, and unstable solutions approach a stable steady state. Meng et al. [25] studied the combined effects of harvesting and time delay on predator–prey systems with Beddington–DeAngelis functional response where the delay is in the predator growth term. Two time delays and predator harvesting are modeled by Zhang et al. [26] to show the stability switching in system dynamics due to time delay. Collera [27] have investigated a delayed model consisting of an intraguild predator and an intraguild prey sharing a common basal resource. A delayed density-dependent term and constant quota harvesting were incorporated in the basal resource. The author found that multiple stability switching occurs at one of the coexisting equilibria. Roy et al. [28] studied the Holling–Tanner model having Beddington–DeAngelis functional response under prey harvesting and gestational delay involved in predation. Pal et al. [29] have studied the effect of delay and harvesting in a predator–prey model with Holling type II functional response and found that an imprecise parameter set plays an important role in the stability. Caraballo et al. [30] have considered different predator–prey models with nonlinear harvesting terms along with constant delay or distributed delay. When the constant (distributed, respectively) delay is varied, they found a stability switching (stability change, respectively) of a coexisting equilibrium. Liu and Huang [31] represented the effect of harvesting in a predator–prey model with Holling type IV functional response. They discussed the bionomic equilibrium, the maximum sustainable total yield (MSTY), and the optimal harvesting strategy into the model. Jana and Kumar Roy [32] have taken into account two types of delays, viz., negative feedback delay of prey and gestation delay of predator in a Holling–Tanner model with Beddington–DeAngelis functional response. The dynamics have been explored due to the varying delays, while harvesting efforts were kept fixed in the nonlinear harvesting terms. Very recently, Majumdar et al. [33] have proved the existence of Bogdanov–Takens bifurcation in predator–prey model subject to Holling type III functional response and nonlinear harvesting strategy.

Several studies have investigated whether harvesting can stabilize (destabilize) a steady state for several population dynamics models having linear as well as Holling type II functional responses [34–36]. It was observed that the results for delayed and non-delayed models differ significantly. A two-patch predator–prey model with delayed population harvesting strategy has been proposed by Pei et al. [37]. They provided several sophisticated mathematical tools and numerical

simulations for achieving optimal yield. It was reported that larger delay can reduce harvested biomass. Recently, Barman and Ghosh [36] studied a class of predator–prey models incorporating time delay in logistic terms of prey growth with linear and Holling type II functional response. They proved that stability of the equilibrium cannot be preserved when time delay is increased. This result contradicts with the earlier work by Martin and Ruan [8]. In addition, they explicitly established the harvesting induced stabilization results. We would like to take this theme forward when time delay is linked with the predation process in different predator–prey systems. Several publications discussed about instability and stability switching in delayed population dynamics models. However, the role of crowding effect and distinct functional responses are not explicitly shown in resulting stability switching and instability. In addition, the validity of delay induced stability switching in other systems involving crowding effects among predators would be addressed. Once delay induced dynamics is analyzed, we study the impact of prey and predator harvesting. Martin and Ruan [8] and Kar and Ghorai [38] have shown that harvesting can stabilize equilibrium, which was unstable prior harvesting. However, they concluded it by computing the time response of the populations for a suitable effort. It does not produce a complete scenario for other efforts. We pay attention to provide the results for the effective range of effort. Moreover, Kar and Pahari [39] noted that increasing effort either on prey or predator enhances the stability of the equilibrium. We attempt to examine if intermediate effort can bring instability. We also verify if stability switching occurs due to harvesting effort in several systems. Some scholarly articles [26,28,30,40,41] proved many mathematical properties of the model under time delay and harvesting, but explicit effects of harvesting were not explained. Therefore, our study makes a more complete analysis addressing the above queries and generates new results in population dynamics models.

This paper is structured as follows: In Section 2, we describe the general model and derive some other ones with appropriate parameter value. Section 3 is devoted to establish the different dynamic modes of the models, with mathematical conditions, by considering time delay as the control parameter. We explore the influence of population harvesting, in the context of stability nature of the equilibrium, for each models, with fixed time delay in the unharvested systems, in Section 4. Finally, we summarize our results, including some comparisons with the earlier literatures, in the last section.

2. Model overview

We consider a predator–prey model with Beddington–DeAngelis functional response [5,6,42] as follows:

$$
\begin{aligned}
\dot{x} &= x\left(a - bx - \frac{cy}{m_1 + m_2 x + m_3 y}\right), \\
\dot{y} &= y\left(-d - \gamma y + \frac{fx(t-\tau)}{m_1 + m_2 x(t-\tau) + m_3 y(t-\tau)}\right),
\end{aligned} \tag{1}
$$

where $x(t)$ and $y(t)$ denotes the prey and predator populations at time t; a and d denote the intrinsic growth rate of the prey and specific mortality rate of the predator, respectively; b and γ are the intraspecies competition coefficients of the prey and predator, respectively; c is the capturing coefficient during predation; and $f = ch$, where h is the conversion coefficient in the predation process. Here, $\tau(>0)$, involved in the numerical response function for the predator, is the time delay in the conversion of prey to predator. We assume that the initial population always satisfies $x(\theta) = \phi_1(\theta) > 0$, $y(\theta) = \phi_2(\theta) > 0$ for $\theta \in [-\tau, 0]$ and $\phi = (\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}_+^2)$. Time delay can occur in the numerical response function in different forms with proper biological meanings ([8,43], and references therein) in population dynamics models. The parameters m_2 and m_3 are linked with handling time for prey capturing by predator and mutual predator interference, respectively. The parameter m_1 is the measure of abundance of prey and predator densities relative to their interacting environment.

Continuous-time model is prefered when populations survive many years and reproduce several times. In our continuous-time model, we incorporated discrete delay. On the other hand, discrete-time models play an important role in population dynamics when species possess non-overlapping generations. Likewise, continuous-time model, discrete delay factors have also been incorporated in discrete-time systems [44–47]. A detailed analytical and numerical bifurcation analysis has been explored by Kot [44] and Yousef et al. [46] when a delayed density dependence is in action among prey species in a predator–prey system. Neverova et al. [45] and Yousef [47] successively studied Moran–Ricker models in the presence of delayed density dependent birth rate to demonstrate very rich and complex dynamics. Interested readers can read more references cited in these articles.

Now, we define four different models without delay as follows:

1. **RM model**: The Rosenzweig–MacArthur predator–prey model is derived from system (1) by considering Holling type II functional response and no intraspecific competition ($\gamma = 0$).

2. **BD model**: The Beddington–DeAngelis predator–prey model is derived from sytem (1) by incorporating Beddington–DeAngelis functional response and no intraspecific competition ($\gamma = 0$).

3. **RMI model**: This is the RM model with intraspecific competition ($\gamma > 0$).

4. **BDI model**: This is the BD model with intraspecific competition ($\gamma > 0$).

Separately, we study the dynamics of the above models under varying time delay and harvesting effort in successive sections.

3. Model dynamics under time delay

This section deals with the stability analysis of the system (1) under varying time delay. We establish the system dynamics around steady state by considering time delay as the control parameter. Such an analysis can be performed by linearizing the delayed nonlinear system around the fixed point.

We will focus on the system where both the species coexist at equilibrium. The conditions on parameters are derived in order to prove the existence of such an equilibrium. In addition, we investigate the model when unique positive equilibrium exists. Let (x^*, y^*) be the interior equilibrium of the system (1), which must satisfy the equations:

$$(a - bx^*)(m_1 + m_2x^* + m_3y^*) - cy^* = 0, \quad (2)$$

$$(-d - \gamma y^*)(m_1 + m_2x^* + m_3y^*) + fx^* = 0. \quad (3)$$

Now, equation (2) yields

$$y^* = \frac{(a - bx^*)(m_2x^* + m_1)}{(c - m_3a) + bm_3x^*}.$$

Then, any of the following conditions needs to be met for the existence of interior equilibrium:

1. **C1**: $c \geq m_3a$ and $0 \leq x^* < \frac{a}{b}$.
2. **C2**: $c < m_3a$ and $\frac{m_3a - c}{bm_3} < x^* < \frac{a}{b}$.
   From equation (3), we get

$$\gamma m_3 y^{*2} + (\gamma m_1 + dm_3 + \gamma m_2 x^*)y^* \\ - [(f - dm_2)x^* - dm_1] = 0,$$

which clearly implies that $y^* > 0$ if $(f - dm_2)x^* - dm_1 > 0$, irrespective of the presence of γ and m_3.

Thus, combining equations (2) and (3), we get, $x^* > 0$ and $y^* > 0$ if and only if $(f - dm_2)a/b > dm_1$.

Since the condition for the existence of interior equilibrium is not affected by the presence of γ and m_3, henceforth, all the four models, viz., RM, BD, RMI and BDI, have interior equilibrium if and only if

$$(f - dm_2)\frac{a}{b} > dm_1.$$

Following [42], the linearized version of the system (1) about the interior equilibrium (x^*, y^*) reads as

$$X' = -px^*X(t) - qx^*Y(t),$$
$$Y' = -ry^*X(t - \tau) - S_1y^*Y(t - \tau) - S_2y^*Y(t), \quad (4)$$

where

$$p = b - \frac{cm_2y^*}{(m_1 + m_2x^* + m_3y^*)^2},$$

$$q = \frac{c(m_1 + m_2x^*)}{(m_1 + m_2x^* + m_3y^*)^2},$$

$$r = \frac{-f(m_1 + m_3y^*)}{(m_1 + m_2x^* + m_3y^*)^2},$$

$$S_1 = \frac{fm_3x^*}{(m_1 + m_2x^* + m_3y^*)^2},$$

$$S_2 = \gamma.$$

The expressions and signs of the above quantities will be different based on models considered. The characteristics equation corresponding to the linearized system is obtained as

$$\lambda^2 + a_1\lambda + a_2\lambda e^{-\lambda\tau} + a_3 + a_4e^{-\lambda\tau} = 0, \quad (5)$$

where

$$a_1 = px^* + S_2y^*,$$

$$a_2 = S_1y^*,$$

$$a_3 = pS_2x^*y^*,$$

$$a_4 = (pS_1 - qr)x^*y^*.$$

The local stability analysis of the nonlinear models can be determined from the equation (5). It is customary to set the system dynamics, without time delay, to a stable steady state in order to establish the delay induced stability changes. Therefore, we first derive the conditions that impose local stability of the equilibrium for the non-delay models.

When $\tau = 0$, equation (5) becomes

$$\lambda^2 + (a_1 + a_2)\lambda + a_3 + a_4 = 0. \quad (6)$$

It is well known results that all four models above, in the absence of time delay, can experience either cyclic dynamics or a globally stable steady state. The necessary and sufficient condition for the local asymptotic stability of the unique equilibrium is

$$a_1 + a_2 > 0 \text{ and } a_3 + a_4 > 0.$$

We assume that the non-delay models are in stable steady state mode and then examine the impacts of varying time delay on system dynamics. If the dynamic behavior is altered with time delay, then

(i) at least one pair of the eigenvalues corresponding to the equation (5) must achieve the form $\pm i\omega$ ($\omega > 0$) for some critical τ, and

(ii) furthermore, a slight increase value of τ can lead positive real part of the said pair(s) of eigenvalues.

We assume that the above situation is possible, and then, $\lambda = i\omega$ must satisfy equation (5). After plugging $\lambda = i\omega$ and by separating the real and imaginary parts, we obtain

$$-\omega^2 + a_2\omega \sin(\omega\tau) + a_4 \cos(\omega\tau) + a_3 = 0, \qquad (7)$$

$$a_1\omega + a_2\omega \cos(\omega\tau) - a_4 \sin(\omega\tau) = 0. \qquad (8)$$

Successively squaring and then summing equations (7) and (8), we get

$$\omega^4 + (a_1^2 - a_2^2 - 2a_3)\omega^2 - a_4^2 + a_3^2 = 0. \qquad (9)$$

Therefore, one can find

$$\omega^2 = \frac{1}{2}\left[(a_2^2 - a_1^2 + 2a_3)\pm \right.$$
$$\left. \sqrt{(a_2^2 - a_1^2 + 2a_3)^2 - 4(a_3^2 - a_4^2)}\right]. \qquad (10)$$

The stability analysis can be established by computing the positive values of ω. Now, we will study explicitly the delayed RM, BD, RMI, and BDI models for its dynamics under varying time delay in the succeeding subsections.

3.1. Delayed RM model

In the RM model (i.e., when $m_3 = \gamma = 0$), the interior equilibrium point of the model can be obtained explicitly and is given by

$$(x^*, y^*) = \left(\frac{dm_1}{f - dm_2}, \left(a - \frac{bdm_1}{f - dm_2}\right)\left(\frac{m_1 f}{c(f - dm_2)}\right)\right).$$

The linearized system about the interior equilibrium is given by equation (4) with $S_1 = S_2 = 0$. Hence, $a_2 = a_3 = 0$. So, the characteristic equation associated with the linearized system reduces to

$$\lambda^2 + a_1\lambda + a_4 e^{-\lambda\tau} = 0. \qquad (11)$$

Henceforth, equation (10) yields

$$\omega^2 = \frac{1}{2}\left[-a_1^2 \pm \sqrt{a_1^4 + 4a_4^2}\right]. \qquad (12)$$

Only one positive $\omega = \omega_+$ can be obtained from (12).

Hence, equation (11) has only one pair of purely imaginary root for some critical value(s) of τ. The values of critical time delay τ_j^+'s can be obtained by substituting ω_+^2 in equation (7), which are calculated as follows:

$$\tau_j^+ = \frac{1}{\omega_+}\arccos\left(\frac{\omega_+^2}{a_4}\right) + \frac{2j\pi}{\omega_+}, j = 0, 1, \dots \qquad (13)$$

Since the equilibrium was asymptotically stable for $\tau = 0$, it maintains the stability for $\tau < \tau_0^+$. At $\tau = \tau_0^+$, a Hopf bifurcation occurs. We would like to know the stability nature of the equilibrium when the time delay increases through $\tau = \tau_0^+$.

To examine if the eigenvalues change their signs for $\tau > \tau_0^+$, we verify the transversality condition,

$$\frac{d}{d\tau}\text{Re}(\lambda)|_{\tau=\tau_j^+} > 0.$$

Differentiating (11) with respect to τ, we obtain

$$(2\lambda + a_1 - a_4\tau e^{-\lambda\tau})\frac{d\lambda}{d\tau} - a_4\lambda e^{-\lambda\tau} = 0. \qquad (14)$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + a_1)e^{\lambda\tau}}{\lambda a_4} - \frac{\tau}{\lambda} \qquad (15)$$

where

$$e^{\lambda\tau} = \frac{-a_4}{\lambda^2 + a_1\lambda}.$$

Now,

$$\text{sign}\left\{\frac{d(\text{Re}\lambda)}{d\tau}\right\} = \text{sign}\left\{\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}$$

$$= \text{sign}\left\{\text{Re}\left[\frac{-(2\lambda + a_1)}{\lambda(\lambda^2 + a_1\lambda)}\right]_{\lambda=i\omega}\right\}$$

$$= \text{sign}\left\{\text{Re}\left[\frac{2\omega^2 - a_1^2}{a_1^2\omega^2 + \omega^4}\right]\right\}$$

$$= \text{sign}\{\text{Re}[2\omega^2 - a_1^2]\} > 0.$$

The last inequality follows from equation (12). It suggests that the same pair of eigenvalues has positive real part when it crosses $\tau = \tau_j^+$. Therefore, the equilibrium is always unstable beyond $\tau = \tau_0^+$.

As a conclusion, we establish that the stable steady state certainly experiences instability when time delay is increased. Neither (i) the coexisting equilibrium remains stable for any $\tau > 0$ nor (ii) the system experiences stability switching under any parameter condition. We acknowledge the work by Martin and Ruan [8] [see their model (3.1)] who found that the coexisting equilibrium remains stable when the time delay is less than a critical threshold and the coexisting equilibrium gets destabilized when the time delay is superior to the critical threshold. However, delay-induced stability switching is impossible to occur.

Interestingly, Li and Takeuchi [42] studied a predator–prey model with Beddington–DeAngelis functional response and intraspecific competition among predators. They incorporated the time delay factor in the numerical response function. They realized that three kinds of dynamics are possible in the delayed model as follows.

(i) There is a situation for which time delay cannot change the asymptotic stability of the equilibrium.

(ii) A suitable condition on parameter space can develop delay-induced instability around the steady state.

(iii) Stability switching can also be observed under certain parameter conditions.

Clearly, two additional stability behaviors (i) and (iii) do not occur in RM model as we have just discussed.

Therefore, one of the following must be responsible for creating the two additional dynamics:
(a) Beddington–DeAngelis functional response,
(b) intraspecific competition among predator,
(c) both (a) and (b)
We are curious to identify the key factor(s) developing such stability modes. We now address this issue by considering the following three models.

3.2. Delayed BD model

First, we only incorporate the Beddington–DeAngelis functional response, which is the delayed BD model ($\gamma = 0$). The linearized system around the coexisting equilibrium will follow equation (4) where $S_2 = 0$. As a consequence, $a_3 = 0$. Thus, the characteristic equation, in this case, reduces to

$$\lambda^2 + a_1\lambda + a_2\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0.$$

Using the similar arguments, as explained earlier, we determine

$$\omega^2 = \frac{1}{2}\left[(a_2^2 - a_1^2) \pm \sqrt{(a_2^2 - a_1^2)^2 + 4a_4^2}\right]. \quad (16)$$

Thus, a positive solution $\omega = \omega_+$ is computed from (16).

This verifies the existence of a pair of imaginary roots of the characteristic equation (5). Substituting ω_+^2 in equations (7) and (8) corresponding to the BD model, we find the values of the critical time delay τ_j^+'s as

$$\tau_j^+ = \frac{1}{\omega_+}\arccos\left(\frac{\omega_+^2(a_4 - a_2 a_1)}{a_4^2 + a_2^2\omega_+^2}\right) + \frac{2j\pi}{\omega_+}, j = 0, 1, \ldots$$

for which Hopf bifurcation appears.

Differentiating the characteristic equation with respect to τ, we obtain

$$\left(2\lambda + a_1 + a_2(1 - \tau\lambda)e^{-\lambda\tau} - a_4\tau e^{-\lambda\tau}\right)\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}$$
$$- (\lambda a_2 + a_4)\lambda e^{-\lambda\tau} = 0.$$

Thus,

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{(2\lambda + a_1)e^{\lambda\tau}}{\lambda(\lambda a_2 + a_4)} + \frac{a_2}{\lambda(\lambda a_2 + a_4)} - \frac{\tau}{\lambda} \quad (17)$$

where

$$e^{\lambda\tau} = \frac{-(a_2\lambda + a_4)}{\lambda^2 + a_1\lambda}.$$

Now,

$$\mathrm{sign}\left\{\frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right\}$$

$$= \mathrm{sign}\left\{\mathrm{Re}\left[\frac{-(2\lambda + a_1)}{\lambda(\lambda^2 + a_1\lambda)}\right]_{\lambda=i\omega} + \mathrm{Re}\left[\frac{a_2}{\lambda(\lambda a_2 + a_4))}\right]_{\lambda=i\omega}\right\}$$

$$= \mathrm{sign}\left\{\mathrm{Re}\left[\frac{2\omega^2 - a_1^2}{a_1^2\omega^2 + \omega^4}\right] + \mathrm{Re}\left[\frac{-a_2^2}{a_2^2\omega^2 + a_4^2}\right]\right\}$$

$$= \mathrm{sign}\left\{\mathrm{Re}\left[2\omega^2 - a_1^2 - a_2^2\right]\right\} > 0.$$

Therefore, when time delay $\tau < \tau_0^+$, the equilibrium is in stable state; a Hopf-bifurcation occurs when $\tau = \tau_0^+$; and instability persists in the system beyond $\tau = \tau_0^+$. We would also like to mention that Liu and Yuan [48] have reported the same mode. Thus, we conclude that exactly the same dynamics occurs in both the RM and BD models under time delay. One might think that invariance of asymptotic stability and stability switching can be an effect of the intraspecific competition term. Therefore, we now study the RM model with intraspecific competition among predators.

3.3. Delayed RMI model

The delayed RMI model is given by the equation (1) with $m_3 = 0$. In this case, $S_1 = 0$, and hence, $a_2 = 0$. The characteristic equation of the system reduces to

$$\lambda^2 + a_1\lambda + a_3 + a_4 e^{-\lambda\tau} = 0.$$

The above equation produces

$$\omega^2 = \frac{1}{2}\left[(-a_1^2 + 2a_3) \pm \sqrt{(-a_1^2 + 2a_3)^2 - 4(a_3^2 - a_4^2)}\right].$$
(18)

We have to determine the signs of the right-hand side expression of the above equation. Here

$$-a_1^2 + 2a_3 = -(px^* + S_2 y^*)^2 + 2pS_2 x^* y^*$$

$$= -(px^*)^2 - (S_2 y^*)^2 < 0.$$

Hence, the positive $\omega = \omega_+$, if exists, can be found from

$$\omega^2 = \frac{1}{2}\left[(-a_1^2 + 2a_3) + \sqrt{(-a_1^2 + 2a_3)^2 - 4(a_3^2 - a_4^2)}\right].$$

The values of the critical time delay τ_j^+'s can be determined by substituting ω_+^2 in equations (7) and (8) and are given by

$$\tau_j^+ = \frac{1}{\omega_+}\arccos\left(\frac{\omega_+^2 - a_3}{a_4}\right) + \frac{2j\pi}{\omega_+}, j = 0, 1, \ldots.$$

Similar to the previous section, we get the transversality condition

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{Re}(\lambda)|_{\tau=\tau_j^+} > 0.$$

Since, there exists at most one positive value of ω, no switching of stability occurs due to time delay. We now verify the remaining two kinds of delay induce dynamics.

(a) If $a_3 > a_4$, no positive ω is obtained. Hence, the interior equilibrium remains stable for any $\tau > 0$.

(b) If $a_3 < a_4$, exactly one positive ω is obtained. Hence, the interior equilibrium alters its stability for some τ, and the revised dynamics mode will persist for further increasing values of τ.

We would like to verify through numerical examples if the above two situations are possible.

Example 3.1. Equilibrium remains stable irrespective of the time delay.

Let us take the parameters $a = 2$, $b = 0.04$, $c = 1$, $d = 3, f = 6, m_1 = 40$, and $m_2 = 1$ with an intraspecific competition coefficient $\gamma = 1$. The interior equilibrium is (49.908, 0.3306), and correspondingly,

$a_1 = 2.3248$, $a_3 = 0.6593$, $a_4 = 0.0054$. Clearly, $a_3 > a_4$, and hence, no positive ω_+ exists. Hence, the unharvested system remains stable for all time delays.

Example 3.2. Equilibrium experiences instability due to time delay

Changing the value of γ in the previous example from $\gamma = 1$ to $\gamma = 0.01$, we get the interior equilibrium as (44.9007, 17.3170). Here $a_1 = 1.8613$, $a_3 = 0.2923$, $a_4 = 0.3049$ ($a_3 < a_4$). We get the roots $\omega^2 = 0.0026, -2.8824$. Hence, only one positive ω exists and a stability change takes place in the model. The critical value of time delay where Hopf bifurcation occurs is calculated to be $\tau_0 = 5.3825$. The coexisting equilibrium remains locally stable for all $\tau < 5.3825$ and remains unstable for all $\tau > 5.3825$. To show the different stability behavior with varying time delay, we have provided two phase diagrams in Figure 1.

The above two examples demonstrate that the conditions, we have proposed, are not absurd. Later, we will recall few of the earlier reports where conditions are stated arbitrarily as they never be satisfied for the investigated models.

The RMI model produces two kinds of stability behaviors. Obviously, the invariance property of asymptotic stability, for all time $t > 0$, is the result of intraspecific competition.

Furthermore, it is observed that when the intraspecific competition is reducing, the delayed RMI model is merging with the RM model, as can be seen from the example above.

Since the above three models do not experience stability switching, we can think that this switching phenomenon is a consequence of Beddington–DeAngelis functional response, or a combined effect of functional response as well as intraspecific competition among predators.

3.4. Delayed BDI model

The delayed BDI model is the given general predator–prey model (1). As before, the characteristic equation is given by (5). Now, we examine the roots of the equation (9) by making use of equation (10), which aregiven as

$$\omega^2 = \frac{1}{2}\left[(a_2^2 - a_1^2 + 2a_3) \pm \sqrt{(a_2^2 - a_1^2 + 2a_3)^2 - 4(a_3^2 - a_4^2)}\right].$$
(19)

We first show that both $a_2^2 - a_1^2 + 2a_3$ and $a_3^2 - a_4^2$ cannot be positive simultaneously. Let us suppose that $a_2^2 - a_1^2 + 2a_3 > 0$ and $a_3^2 - a_4^2 > 0$. Then,
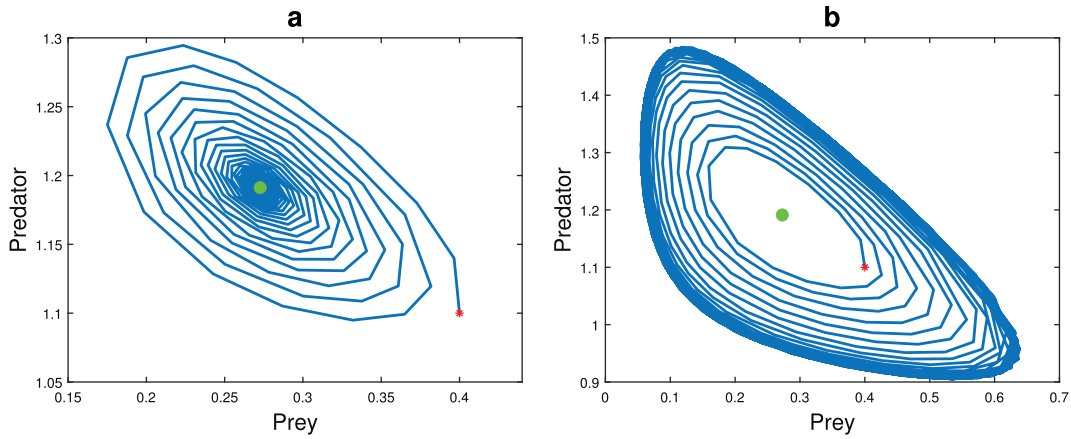
Figure 1. We assume the parameter set to be $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$, $m_2 = 0.2$ and $\gamma = 0.01$. The critical time delay is found to be $\tau_0 = 5.3825$. (a) For $\tau = 5 < \tau_0$, the initial prey–predator $(0.4, 1.1)$ (indicated in red star) tends towards the coexisting equilibrium $(0.2727, 1.1911)$. (b) On the other hand, the same initial condition moves towards a limit cycle, and hence, the coexisting equilibrium $(0.2727, 1.1911)$ (indicated with green dot) becomes unstable when $\tau = 6 > \tau_0$.

$$a_2^2 - a_1^2 + 2a_3 = (S_1 y^*)^2 - (px^* + S_2 y^*)^2 + 2pS_2 x^* y^*$$

$$= (y^*)^2 (S_1^2 - S_2^2) - (px^*)^2 > 0$$

yields $(y^*)^2 (S_1^2 - S_2^2) > (px^*)^2$.

Hence $S_1 > S_2$.

But

$$a_3^2 - a_4^2 = (pS_2 x^* y^*)^2 - (pS_1 - qr)^2 (x^* y^*)^2 > 0 \quad (20)$$

gives $(pS_2)^2 > (pS_1 - qr)^2,$ \quad (21)

which is a contradiction since $S_1 > S_2$ with $r < 0$.

This ensures the non-existence of two positive ω. Henceforth, the switching of stability is not possible in this model. We now show that other two delay-induced dynamics are possible in this general model under the following conditions:

(a) If $a_2^2 - a_1^2 + 2a_3 < 0$ and $a_3^2 - a_4^2 > 0$, no positive ω is obtained. Hence, the interior equilibrium remains stable for any $\tau > 0$.

(b) If $a_3^2 - a_4^2 < 0$, exactly one positive ω is obtained. Hence, the interior equilibrium alters its stability for some τ and the revised dynamic mode will persist for further increasing values of τ.

The corresponding critical time delays for the positive ω_+ are given by

$$\tau_j^+ = \frac{1}{\omega_+} \arccos\left(\frac{a_4(\omega_+^2 - a_3) - a_1 a_2 \omega_+^2}{(a_2^2 \omega_+^2) + a_4^2} \right) + \frac{2j\pi}{\omega_+}, j = 0, 1, \ldots$$

We verified that the transversality condition yields a positive sign for the above critical delays.

An equality condition $(a_2^2 - a_1^2 + 2a_3)^2 = 4(a_3^2 - a_4^2)$ along with $a_2^2 - a_1^2 + 2a_3 > 0$ is stated by [42] for the occurrence of instability. However, our condition for instability is slightly different. The equality condition does not hold because $a_3^2 - a_4^2$ and $a_2^2 - a_1^2 + 2a_3$ cannot have positive sign simultaneously. Also some researchers might think that existence of two positive ω's induces stability switching. Recent investigation by Barman and Ghosh [36] showed that two positive ω's may produce only instability, but not stability switching.

The above information, for the BDI model, can be illustrated by numerical examples as follows .

Example 3.3. Equilibrium remains stable irrespective of the time delay.

Taking the parameter set $a = 1$, $b = 4.6$, $c = 1.5$, $d = 0.05$, $\gamma = 2$, $f = 1.5$, $m_1 = 1$, $m_2 = 1$, and $m_3 = 0.2$, we get the interior equilibrium $(0.1920, 0.0939)$. The numerical values of $a_1 = 1.0530$, $a_2 = 0.0036$, $a_3 = 0.1625$, $a_4 = 0.02613$. Here, $a_2^2 - a_1^2 + 2a_3 = -0.7837 < 0$ and $a_3^2 - a_4^2 = 0.0257 > 0$ are satisfied as mentioned in (a). Henceforth, we get two negative values, $\omega^2 = -0.0343, -0.7493$. Thus, the system remains stable for any value of time delay .

Example 3.4. Equilibrium experiences instability with unique positive ω

Taking the same parameter set as in the previous case, except the change of $\gamma = 2$ to $\gamma = 0.1$, we get the interior equilibrium as $(0.0787, 0.5014)$ and the numerical values of $a_1 = 0.3696$, $a_2 = 0.0085$, $a_3 = 0.0160$, $a_4 = 0.0572$. Here, $a_3^2 - a_4^2 = -0.003 < 0$. Thus, we get the values of the two
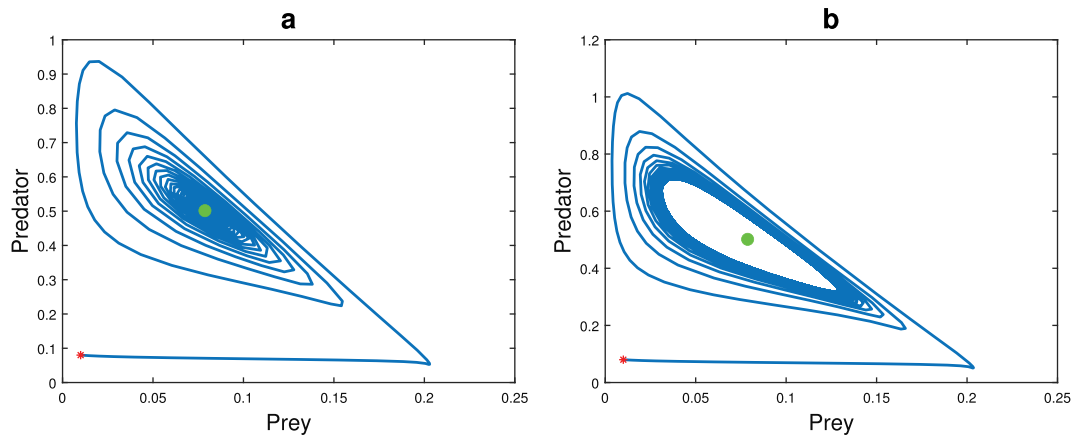
Figure 2. Under the parameter setting $a = 1$, $b = 4.6$, $c = 1.5$, $d = 0.05$, $\gamma = 0.1$, $f = 1.5$, $m_1 = 1$, $m_2 = 1$, and $m_3 = 0.2$, we calculated $\tau_0 = 9.5103$. (a) The trajectory starting with the initial condition $(0.01, 0.08)$ (indicated with red star) tends towards the coexisting equilibrium $(0.0787, 0.5014)$ when $\tau = 9 < \tau_0$. (b) However, the trajectory starting with the same initial condition converges towards the limit cycle for $\tau = 10 > \tau_0$. Hence, the equilibrium (indicated with green dot) is unstable.

ω^2 as 0.0235 and -0.1281, which implies that stability change exists in the system. The critical value of time delay where Hopf bifurcation occurs is $\tau_0 = 9.5103$. We provide the phase portrait diagrams in Figure 2 for two different values of time delay.

By and large, we observed that RM and BD models only produce instability for some critical time delay. However, RMI and BDI models not only induce instability, but also can maintain asymptotic stability of the equilibrium for any $\tau > 0$. Either model, we have proposed, does not show stability switching under any parameter space.

4. Dynamics of delayed model under harvesting

We established that a system may be either at stable steady state irrespective of the time delay, stable for smaller values of time delay or in unstable mode for relatively larger time delay. We would like to examine the change in dynamic mode with a fixed time delay but varying harvesting rate.

The delayed predator–prey model with Beddington–DeAngelis functional response under harvesting of both the species is given as

$$\begin{aligned} \dot{x} &= x\left(a - bx - \frac{cy}{m_1 + m_2 x + m_3 y}\right) - E_1 x, \\ \dot{y} &= y\left(-d - \gamma y + \frac{fx(t-\tau)}{m_1 + m_2 x(t-\tau) + m_3 y(t-\tau)}\right) - E_2 y, \end{aligned} \quad (22)$$

where E_1 and E_2 are the efforts of prey and predator harvesting, respectively. For simplicity, we assumed the harvesting coefficients as unity. Instead of setting the efforts in the model (1), one can explore the harvesting results by reducing (increasing, respectively) the value of a (d, respectively) for prey (predator, respectively) harvesting. However, parameters including a and d in the ecological model have intrinsic property. Hence, we should vary the efforts as the control parameters. Therefore, setting such efforts in any ecological model is biologically justified.

First, we investigate the system dynamics under prey harvesting for all the four models followed by predator harvesting, fixing the time delay constant.

4.1. Prey harvesting

In this case, $E_2 = 0$ and the prey harvesting effort lies between 0 and $(a - bdm_1/(f - dm_2))$ for coexistence of interior equilibrium. All the different scenarios for the four different models are given in the succeeding subsections.

4.1.1. Prey harvesting in RM model

The RM model always produces non-equilibrium dynamics under time delay. We would examine if harvesting can stabilize (destabilize, respectively) the steady state when the unharvested system with time delay is at non-equilibrium (stable steady state, respectively) mode. For arbitrary effort on prey species, we can obtain the similar expression (as discussed for RM model) of the critical value of time delay causing Hopf bifurcation. It seems impossible to find the threshold values of E_1 for

Hopf bifurcation by fixing the time delay as constant because no explicit value of effort can be determined from equation (13). Therefore, for each fixed effort, we compute critical time delay τ_0 for understanding harvesting influences. Let us take $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$, and $m_2 = 0.2$ for further illustration of the harvesting impacts. Then, the effort of harvesting will lie in the range $[0, 0.9)$ in order to maintain species coexistence. With the help of equation (13), we found the variation of $\tau_0(E_1)$ in the $E_1 - \tau$ plane. Figure 3(a) shows two region separated by $\tau_0(E_1)$. The lower (upper, respectively) one is the region of stability (instability, respectively). We have verified that $\tau_0(E_1)$ is an increasing function of E_1 and has asymptote at $E_1 = 0.9$. The existence of such an asymptote can be visible if the curve is drawn for an extended τ-axis. We observe

that harvesting does not alter asymptotic stability when the time delay in the unharvested model is very small (for example, $\tau = 1$). On the other hand, harvesting always stabilizes the non-equilibrium dynamics, which was in unstable mode in the unharvested system due to time delay. Because of the asymptotic nature of $\tau_0(E_1)$, it is always possible to stabilize the system even if the delay is larger in the unharvested system. For a better explanation, we draw phase portraits corresponding to Figure 3(a) in understanding the stability change due to harvesting. As discussed earlier, we keep $\tau = 8$ fixed (delay is an intrinsic parameter) and vary only the effort. We choose two coordinates $(E_1, \tau) = (0.1, 8)$ and $(E_1, \tau) = (0.6, 8)$, which belong to unstable and stable regions in Figure 3(a), respectively. In Figure 4(a) [Figure 4(b), respectively], it is shown that the
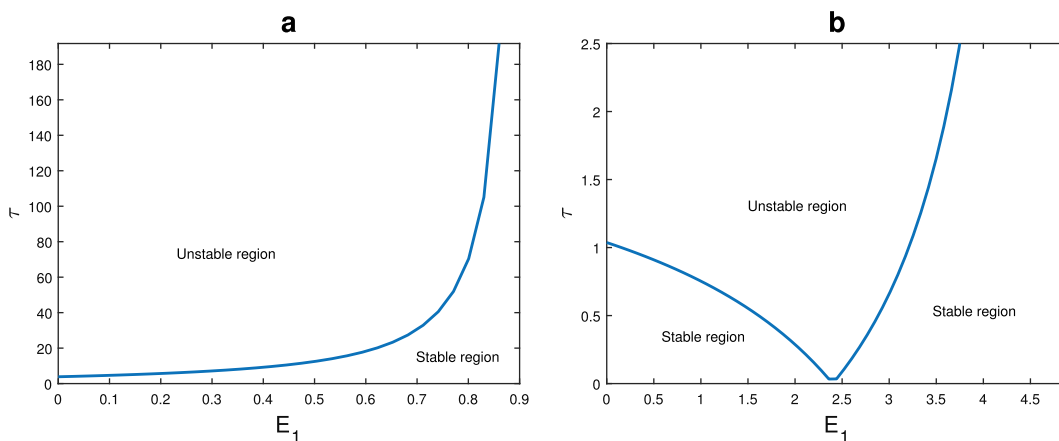
Figure 3. The common parameters for generating the figures are chosen as $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$ and $m_2 = 0.2$. (a) Taking $a = 1$, the variation of τ_0 with respect to effort is shown. (b) With $a = 5$, we have plotted $\tau_0(E_1)$, which decreases for smaller effort and then increases for relatively larger effort.
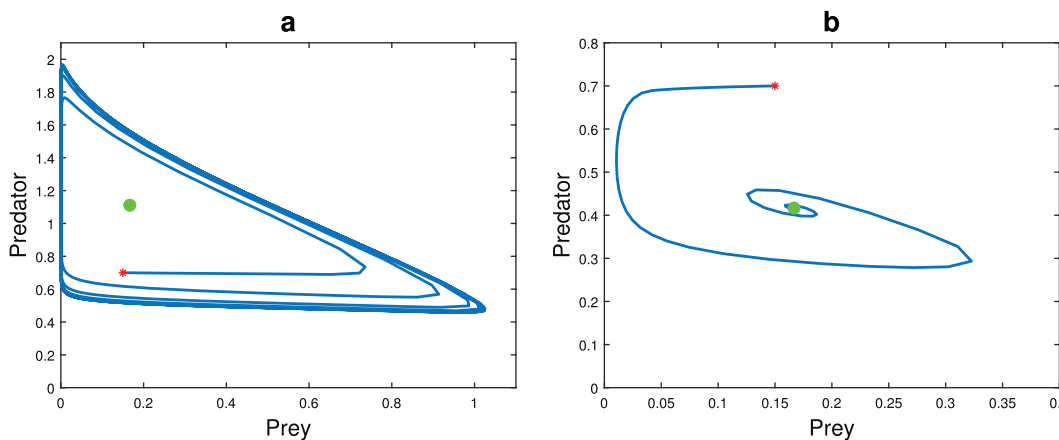
Figure 4. The parameters are the same as those we considered for Figure 3(a) with $\tau = 8$. (a) The trajectory starting at the initial condition $(0.15, 0.7)$ (indicated in red star) moves away from the coexisting equilibrium $(0.1666, 1.1111)$ (indicated in green dot) and converges towards a limit cycle when $E_1 = 0.1$. (b) When $E_1 = 0.6$, the solution with the same initial condition $(0.15, 0.7)$ tends towards the equilibrium $(0.1666, 0.4166)$.

coexisting equilibrium is an unstable focus (a stable focus, respectively) when $(E_1, \tau) = (0.1, 8)$ $[(E_1, \tau) = (0.6, 8)$, respectively].

We would like to recall the result by Martin and Ruan [8] who suggested that harvesting of prey can stabilize the system in an RM type model if the equilibrium is unstable due to time delay. However, they concluded it by computing the trajectories of the model ODEs for a particular harvesting quota. Similar strategy with constant effort is also adapted by Kar and Ghorai [38] in a predator–prey model. However, we have a more general approach that will help in determining the dynamics mode for any harvesting effort. Fortunately, the outcomes by Martin and Ruan [8] and Kar and Ghorai [38] are correct in their respective models, but the techniques might not address the correct results for other situations. Based on their reports, one might think that a stable system remains stable under harvesting. Yes, it is certainly true as can be seen from Figure 3(a). However, the set of parameters with changed value of $a = 5$ produces the curve $\tau_0(E_1)$ which is a decreasing function for smaller effort [see Figure 3(b)]. In this case, the effort of harvesting lies in the range $[0, 4.9]$. If the equilibrium of the unharvested system is stable for smaller delay ($\tau = 0.5$), smaller harvesting effort can destabilize the equilibrium and further increase of effort can stabilize the dynamic mode. Hence, a switching of stability occurs due to prey harvesting.

It is interesting to note that Barman and Ghosh [36] concluded that harvesting cannot stabilize a system when the non-equilibrium dynamics occurs for larger time delay prior to harvesting. They developed the model by incorporating time delay in the logistic prey growth. The region of stability in their analysis was bounded. However, in this case, the region of stability is unbounded due to the asymptotic nature of $\tau_0(E_1)$, which leads to harvesting induced stability. Thus, influences of harvesting depends upon the nature of the unharvested system.

4.1.2. Prey harvesting in BD model

Like the RM model, the BD model also experiences the similar dynamics due to time delay. We would be interested to verify if harvesting induces similar stability behavior in the BD model, which was observed in the RM model. The same approach, we explained for the RM model, can be adopted to analyse the BD model as well for realizing the harvesting influences. We select the same parameter set that was used to generate Figure 3(a) for the RM model with varying m_3. It is to be noted that the maximum limit of effort $E_1 = 0.9$, in order to population persistence, does not depend upon the values of m_3. We perform the

experiments for $m_3 = 0.3, 0.8$ and 3, which generate three curves for $\tau_0(E_1)$ in Figure 5(a,b,c), respectively. Each curve divides the $E_1 - \tau$-plane into two parts: the lower one is stable and the upper one is unstable. It is to be noted that smaller and larger values of m_3 produce similar kind of curve, which are increasing with effort. However, the curve has a valley when m_3 takes some intermediate value. Clearly, a stable state incorporating time delay in the unharvested system does not change its dynamics when prey is harvested under certain parameter space [see Figure 5(a,c)]. However, Figure 5(b) shows that a stable steady state can achieve instability for some intermediate effort, and then, further increase in effort can stabilize the equilibrium. Hence, a stability switching occurs due to effort. We recall that no stability switching is possible due to time delay in RM and BD models. Therefore, the influence of harvesting effort and time delay should be studied explicitly. We have noticed that the influence of prey harvesting differs between RM and BD models.

4.1.3. Prey harvesting in RMI model

Unlike the delayed RM and BD models, the RMI model experiences two types of stability nature under varying τ. The coexisting equilibrium may either be (i) stable for all τ or (ii) stability change occurs beyond some critical τ. We investigate if harvesting have any effect on the nature of the system when τ is fixed in the unharvested system. We take the first parameter set $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$ and $m_2 = 0.2$, which was chosen for the RM model. Intraspecific competition coefficient γ is varied to establish two kinds of dynamics of the RMI model before harvesting. It is observed that, for $0 < \gamma < 0.1$, the quantity $a_3 < a_4$ (see the expression in Example 3.2), which ensures the existence of a positive ω. Thus, there is a critical value of τ below which the coexisting equilibrium of the system remains stable and beyond which the coexisting equilibrium becomes unstable. A Hopf bifurcation occurs at the critical delay. When γ is increased beyond $\gamma = 0.1$, the quantity $a_3 > a_4$, which indicates the absence of positive ω. Hence, the unharvested system remains stable irrespective of the delay parameter. The consequences of harvesting on both the cases are now addressed separately.

Case I: When the unharvested system undergoes delay induced bifurcation

Since for $\gamma < 0.1$, the system undergoes stability change due to time delay, we take $\gamma = 0.05$ to observe the dynamics under harvesting. The nature of the curve $\tau = \tau_0(E_1)$ is shown in Figure 6(a). The harvesting impacts, here, are similar to the ones found in the RM model. However, it is observed that same harvesting
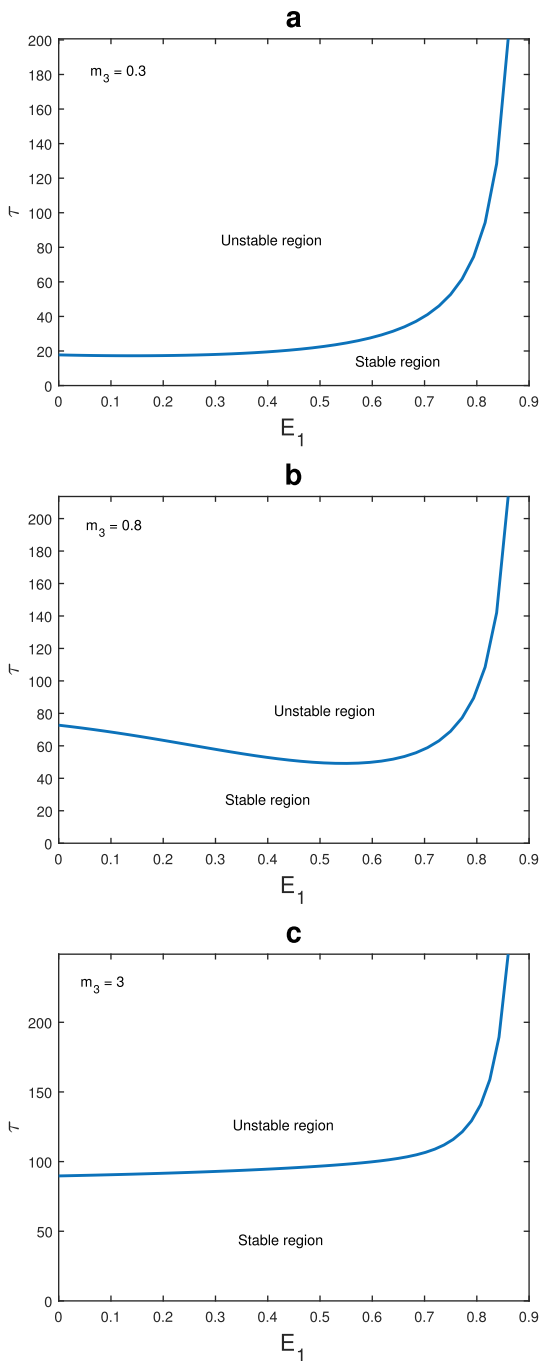
Figure 5. The variation of τ_0 with respect to effort is shown. The parameters are chosen as $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$, and $m_2 = 0.2$ with (a) $m_3 = 0.3$, (b) $m_3 = 0.8$ and (c) $m_3 = 3$. The qualitative nature of $\tau_0(E_1)$ is the same for (a) and (c), whereas the curve in (b) has a unique minimum. The existence of a minimum makes the difference in dynamics when the prey is harvested.

effect still persists for any values of $\gamma < 0.1$. We also investigate if any different nature can be observed with the change in the parameter set. We replace $a = 5$ and

$\gamma = 0.01$ in the existing parameter set. Calculation yields that $a_3 < a_4$ in the unharvested system. Figure 6 (b) depicts the dynamics with varying effort, which produces a stability switching under harvesting.

*Case II: When the unharvested system remains stable for all time delay*

Here, we focus on the case when $\gamma > 0.1$. Since the equilibrium of the unharvested system is asymptotically stable irrespective of time delay, a natural question arises whether stability is preserved irrespective of harvesting effort too.

Taking the intra-specific competition co-efficient $\gamma = 0.12$ in the unharvested system, we observe that $a_3 > a_4$. So, the unharvested system remains stable irrespective of the time delay. With a change in the effort of harvesting, the coexisting equilibrium does not maintain its stable behavior for all τ. For some τ (say $\tau = 100$), the system maintains its stability for all effort, but there exists some range of τ ($\tau = 500$ belongs to the range) for which stability switching takes place (see Figure 7). Thus, harvesting induces stability switching for some range of delay.

However, when we take $\gamma = 0.3$, the unharvested system yields two negative ω^2's. The introduction of harvesting does not change the sign of the two ω^2's. Hence, harvesting has no influence on altering the stability of the system under any constant time delay. It also suggests that a larger intraspecific competition coefficient is favourable to preserve the stability of the equilibrium.

4.1.4. Prey harvesting in BDI model

Two typical scenarios under harvesting, as found for the previous three models, are also possible for the BDI model (see Figure 8). We would be more interested in the situation when the coexisting equilibrium in the unharvested system is stable irrespective of the time delay. Taking the parameter set as $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $f = 0.1$, $m_1 = 0.8$, $m_2 = 0.2$, and $m_3 = 0.3$ (parameters taken for BD model) along with $\gamma = 0.09$, we get a stable equilibrium of the unharvested system for all τ. However, harvesting can bring about stability switching for some range of τ, which is similar to the case found in the RMI model (see Figure 7). By choosing $\gamma = 1$, we find that the equilibrium of the unharvested system remains stable for all τ. Further change in effort does not change the stability nature of the coexisting equilibrium. Hence, the system remains stable for all effort under any fixed time delay.
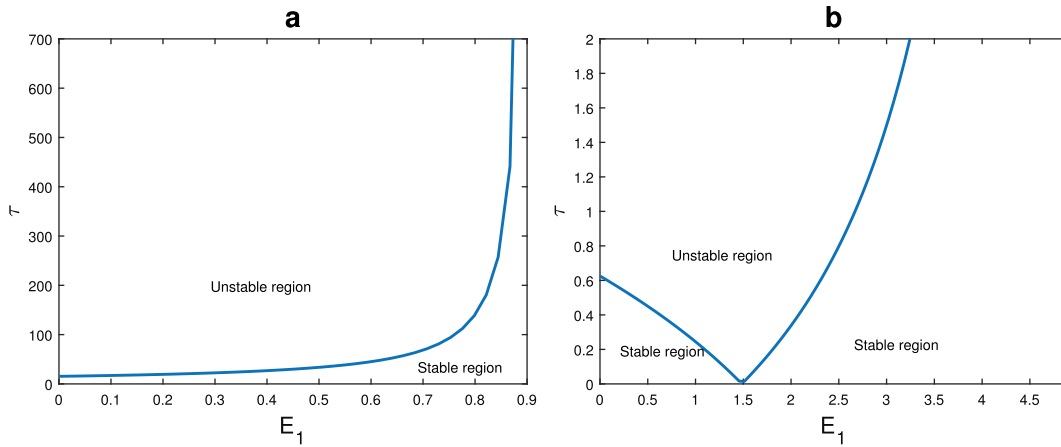
Figure 6. The curves $\tau = \tau_0(E_1)$s are shown as functions of effort. (a) The parameter set is the same as the parameters taken for Figure 3(a) along with $\gamma = 0.05$. (b) It is to be noted that the curve does not touch the E_1 axis. The parameter set is same as the parameters used for Figure 3(b) along with $\gamma = 0.01$.
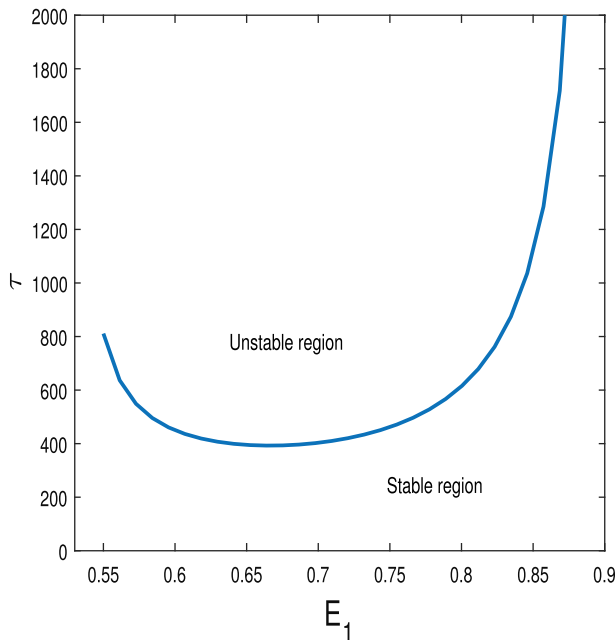
Figure 7. The system remains stable up to some effort for smaller values of τ. When the effort is increased, the system experiences stability switching.

4.2. Predator harvesting

In the case of predator harvesting, we set $E_1 = 0$ in the system (22). The effort of harvesting E_2 follows the inequality:

$$0 < E_2 < \frac{fa}{m_2 a + bm_1} - d,$$

for the existence of interior equilibrium.

Like the prey harvesting, we investigate the system dynamics under predator harvesting for each model. It can be seen that many results are qualitatively similar to the ones we obtain for prey harvesting. Therefore, we will briefly explain and present the results here.

4.2.1. Predator harvesting in RM model

To study the influence of predator harvesting, we recall the same parameter sets taken for the RM model under prey harvesting. The effort of predator harvesting lies in the range $(0, 0.1271)$ for the parameter set associated with Figure 3(a). This parameter set generates Figure 9 (a). On the other hand, the effort range becomes $(0, 0.3178)$ corresponding to the parameter set chosen for earlier Figure 3(b). The curve $\tau = \tau_0(E_2)$ depicting the stability and instability regions is given in Figure 9 (b). Thus, we can obtain both the dynamic modes for predator harvesting, which were found for prey harvesting too.

### 4.2.2. Predator harvesting in BD model

We take the same parameter sets taken previously for the prey harvesting in the BD model to compare our results with the predator harvesting. The curve $\tau = \tau_0(E_2)$ in Figure 10(a,b) depicts two different natures of two different values of m_3.

### 4.2.3. Predator harvesting in RMI model

The effects of predator harvesting are categorized into two cases:

*Case I: When the unharvested system experience stability change with varying time delay*

A similar phenomenon can be observed under predator harvesting as explained for prey harvesting. For a lower range of time delay, effort of harvesting brings
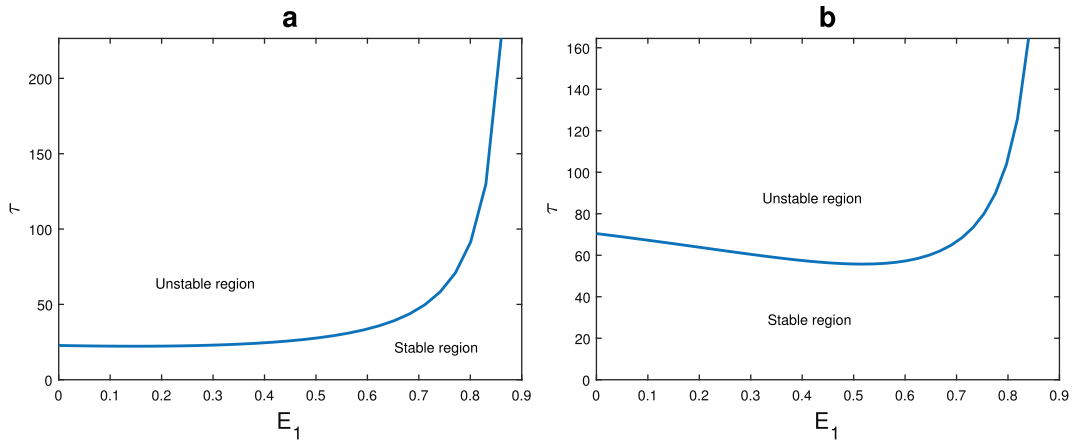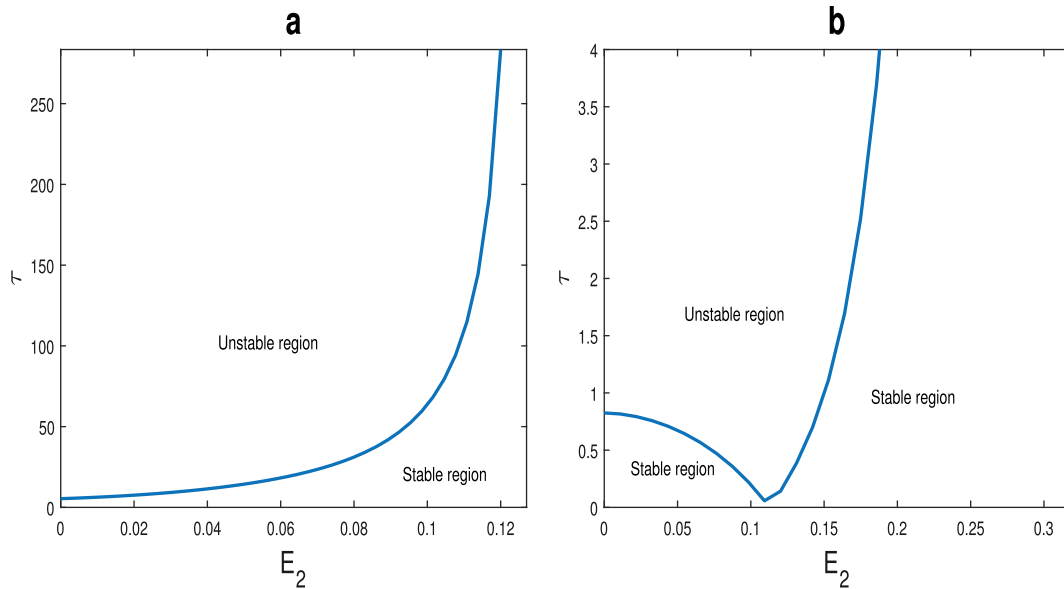
Figure 8. (a) The parameter set is the same as the parameters taken for Figure 5(a) along with $\gamma = 0.01$. (b) The parameter set is the same as the parameters used for Figure 5(b) along with $\gamma = 0.01$.
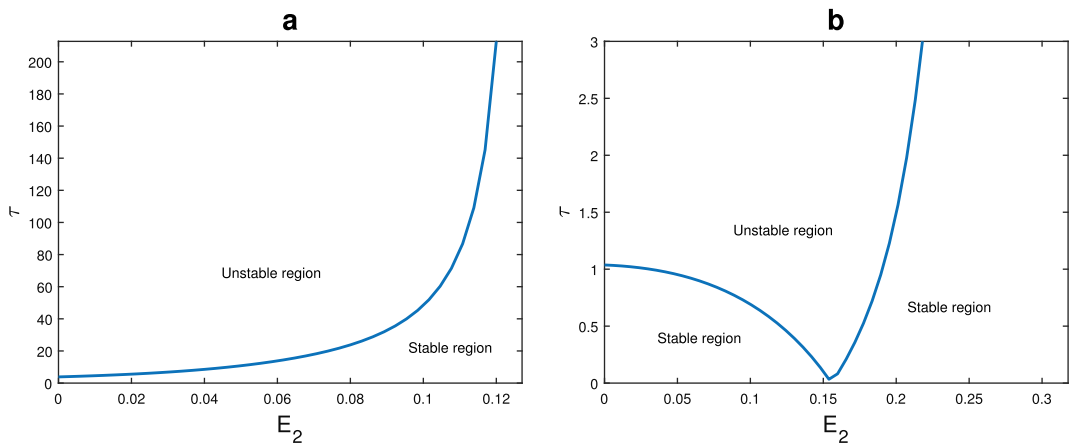
Figure 9. (a) The parameter set is the same as the parameters taken for Figure 3(a). Stable equilibrium stays stable or unstable equilibrium can be stabilized under predator harvesting. (b) The parameter set is the same as the parameters used for Figure 3(b). Stability switching may occur under harvesting.
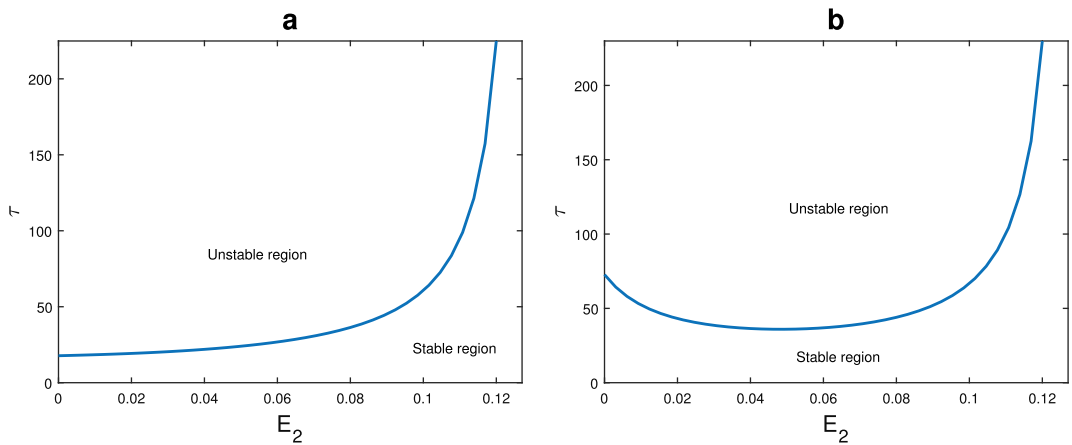
Figure 10. (a) The parameter set is the same as the parameters associated with Figure 5(a). Stability change takes place in the system under harvesting for some fixed higher value of τ. (b) The parameter set is the same as the parameters used for Figure 5(b). Stability switching occurs under harvesting.

Figure 11. (a) The parameter set is the same as the parameters taken for Figure 3(a) along with $\gamma = 0.01$. The unstable equilibrium of the system with larger time delay (for example, $\tau = 20$) can be stabilised by some suitable choice of harvesting effort. (b) The parameter set is $a = 5$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $\gamma = 0.005$, $f = 0.1$, $m_1 = 0.8$, and $m_2 = 0.2$ with effort in the range (0,3178). In this case, a stability switching occurs due to harvesting when the fixed time delay is relatively smaller (for example $\tau = 0.5$).

about no change in the stability of the equilibrium [see Figure 11(a)]. The unstable equilibrium in the unharvested system due to relatively larger time delay can be stabilized by some effort. However, it can be observed in Figure 11(b) that the system, incorporating intermediate delay, experiences stability switching under harvesting.

Case II: When the unharvested system remains stable for all time delay

The equilibrium for the parameter set $a = 1$, $b = 0.6$, $c = 0.6$, $d = 0.02$, $\gamma = 0.3$, $f = 0.1$, $m_1 = 0.8$, and $m_2 = 0.2$ does not have any effect on changing the stability due to time delay. We find that the effort of harvesting, lying in the range (0, 0.1271), does not alter the stability of the system. Even if $\gamma = 0.12$, predator harvesting does not destabilize the equilibrium. We noticed that for the same parameter set, prey harvesting caused stability switching (see case II in Subsection 4.1.3). Therefore, prey and predator harvesting may have very different impacts in predator–prey models.

4.2.4. Predator harvesting in BDI model

The effects of predator harvesting, for the case where the unharvested system experiences stability change due to time delay, are similar under predator harvesting in the RMI model. Hence, we are not providing the details here. We are now interested in the case where the unharvested system is stable for all time delays.
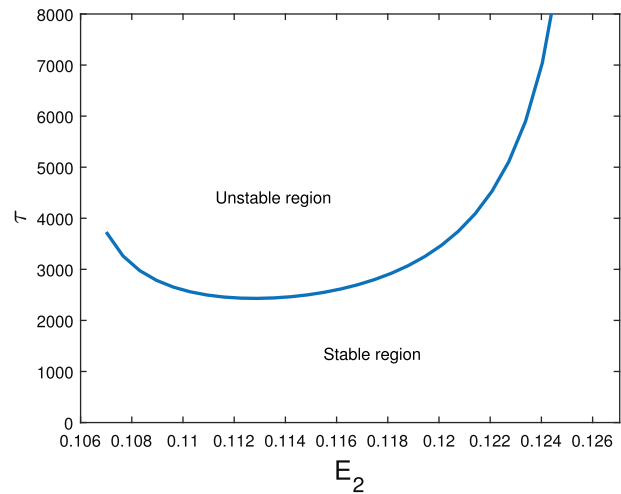
Figure 12. The parameter set is the same as the parameters taken for Figure 5(a) along with $\gamma = 0.09$. For $\tau = 2000$, no effort of harvesting can destabilize the equilibrium. However, when $\tau = 3000$, harvesting induces a stability switching.

Figure 12, depicts the case where the coexisting equilibrium is stable prior to harvesting. Obviously, predator harvesting, in the present situation, can bring about stability switching when time delay $\tau = 3000$ fixed. It can be noted that for $\gamma = 0.1$ in the parameters taken, the system has no influence of harvesting on its stable state.

5. Discussion

In this paper, we have revealed several dynamics in a class of population dynamics models caused by time delay and harvesting strength. We have identified the ecological factors involved in models producing variety in the results. Four different models, viz., RM, BD, RMI, and BDI models, incorporating the time delay in the functional response were analyzed. First, we focused on exploring delay induced dynamics in the systems without harvesting activity. It was found that the stable steady state of the RM model certainly experiences instability when the time delay is increased. This result synchronizes with the one stated by Martin and Ruan [8]. Next, we have proved that the same dynamics occurs in the BD model under changing time delay. It indicates that Beddington–DeAngelis functional response in the BD model does not have any influence on the production of distinct dynamics compared to the Holling-type functional response in the RM model. However, the delay has two different effects on the RMI model:

1. Under certain parameter space, the equilibrium is always stable for all time delay $\tau > 0$ (see Examples 3.1).

2. The model may exhibit delay-induced instability (see Examples 3.2).

Therefore, the delay-induced outcomes are not exactly similar with RM and BD models. We can now claim that the invariance asymptotic stability property of the equilibrium for all delay values is the influence of the crowding factor (density dependent mortality) among predators. We have also reported similar observation for the BDI model as well.

Li and Takeuchi [42] suggested that stability switching may occur in the BDI model. We have analytically proved that no switching of the stability phenomenon is possible in the BDI model. We would like to recall that delay in the logistic prey growth function can generate stability switching in population dynamics models (see [8,36] and references therein). Here, we have established that, when the delay is incorporated in the numerical response, no switching phenomenon can be noticed under varying time delay.

Second, we moved on to investigate the dynamics when prey harvesting is imposed on the models with time delay kept fixed. RM and BD models showed one of the following dynamic modes:

1. The stable equilibrium of the unharvested system with smaller time delay stays stable under harvesting.

2. Harvesting causes switching of stability in the system with intermediate values of time delay.

3. Unstable system involving larger time delay can be stabilized for increasing effort.

Similar observations were also detected in RMI and BDI models under prey harvesting. However, persistence of stability for all time delay is also possible for these two models, prior to harvesting. The impact of prey harvesting was discussed for this situation. We found a parameter set where harvesting does not change the asymptotic stability of the equilibrium irrespective of the time delay in the unharvested model (see Case II of Subsection 4.1.3). Also, stability switching may appear due to harvesting (see Figure 7). We would like to emphasize that the first situation, i.e., the invariance stability property irrespective of delay and harvesting, was not observed in predator–prey models involving time delay in logistic prey growth [36].

In all the models, larger effort on prey species always stabilized the equilibrium, regardless of the value of time delay in the unharvested system, as the $\tau = \tau_0(E_1)$ curve has an asymptote. It was not certain to stabilize the equilibrium under prey harvesting, when the time delay was present in the logistic growth function, as discussed by Barman and Ghosh [36]. There exist situations in which the system remains stable for all effort, but harvesting brings about switching of stability. In all the cases of prey harvesting, it ascertains that harvesting effort ultimately leads to stability in the system.

To gain complete knowledge, we further examine the effects of predator harvesting. Like the prey harvesting, predator harvesting causes three possible dynamics: (i) stable equilibrium stays stable, (ii) unstable equilibrium alters the stability nature and (iii) switching of stability takes place. In all models, equilibrium becomes stable for a sufficiently large effort in predator harvesting as well. However, it turns out that predator harvesting in the RMI model does not always have the same impact found for prey harvesting. For example, Case II in Subsection 4.2.3 showed that predator harvesting does not have any effect in destabilizing the equilibrium, but prey harvesting under the same set of parameters causes stability switching. Henceforth, it indicates that prey and predator harvesting do not always produce the same impacts under same natural conditions of the unharvested system. Kar and Pahari [39] reported that when harvesting effort is increased either on the prey or predator in the RM model, the stabilizing effect is also increased. We have identified a parameter space where the outcome could be different [see Figure 3(b) and Figure 9(b)]. Ghosh et al. [34] have stated that in a non-delayed RM model, predator harvesting cannot induce stability switching. However, we have found a parameter space for which the delayed RM model undergoes a stability switching under predator

harvesting. Overall, scrutinizing all the possible circumstances under both prey and predator harvesting, it can be concluded that regardless of the population being harvested or the value of time delay, harvesting brings about stability in the system before the critical effort is reached. As an additional information, one may consider the combined harvesting effort ($E_1 = E_2 = E$) to examine the impact of harvesting. In this case, the resultant dynamics under harvesting could be similar with the dynamics of either prey-oriented or predator-oriented harvesting. Therefore, the obtained results under a joint harvesting are expected to belong to one of the dynamical behaviors we have demonstrated earlier for individual harvesting strategy.

6. Conclusion

In all the four models, time delay is incorporated in the numerical response function. We conclude that time delay always destabilizes the coexisting equilibrium in RM and BD models, while the equilibrium maintains its local stability in RMI and BDI models. We have identified that the latter outcome in RMI and BDI models is the influence of strength of intra-specific competition among predators. Analytically, we also proved that delay-induced stability switching never occurs in either model when time delay is present in numerical response function. We have uncovered the impacts of harvesting in the delayed system. It is observed that harvest-induced stability results may be different for prey and predator exploitations. However, relatively larger effort on prey or predator must stabilize the equilibrium. Although time delay could not exhibit stability switching, harvesting of prey or predator may have potential to produce stability switching.

The stability switching phenomenon is commonly experienced by many population models with interesting ecological interpretations. Very recently, Bajeux and Ghosh [49] have explored the hydra effects and stability switching in a two-patch continuous predator–prey system without time delay. Stability switching occurs for varying delay when a single discrete delay is incorporated in logistic growth for prey species [21,36] or in dispersal of populations [14,50]. Population models with multiple delays [51] often causes stability switching. In our current study, time delay in numerical response could not produce any stability switching. However, we have considered a limited number of functional/numerical response functions. Therefore, a question must arise whether stability switching and many more delay-induced complex dynamics are possible due to a single time delay with some other proposed functional/numerical response functions in ecology.

As a future perspective, it is important to identify such predator–prey models with a single time delay to unveil new and interesting delay-induced dynamics.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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