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3 (Sem–1/CBCS) MAT HC 2

2020

(Held in 2021)

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(a) Find the polar representation of the point $(2,-2)$.

(b) Find the Cartesian co-ordinates of the point $\left(2, 2\pi/3\right)$.

(c) If $f : R \to R$ is given by $f(x) = x^2$, what is $f^{-1}((0, 4))$?

Contd.

(d) Write the statement and its negation using quantifiers.
"In each tree in the garden, we can find a branch on which all of the leaves are green".

(e) If A is the set of all $n \times n$ symmetric matrices and B is the set of all $n \times n$ real skew-symmetric matrices, what is $A \cap B$?

(f) Let $M(2, \mathbb{R})$ denote the set of all 2×2 matrices over $\mathbb{R}$.
Consider the function $f : M(2, \mathbb{R}) \to \mathbb{R}$ given by $f(A) = \det A$.
Show that f is not one-one.

(g) State the well-ordering principle in $\mathbb{N}$.

(h) *State 'true'* **or** *'false' with justification :*
If one row in an echelon form of an augmented matrix is $[0\ 0\ 0\ 5\ 0]$, then the associated linear system is inconsistent.

(i) *State 'true'* **or** *'false' with justification :*
Each column of AB (where A and B are matrices whose product AB is defined) is a linear combination of the columns of B using weights from the corresponding columns of A.

(j) *Fill in the blanks :*
If A is a triangular matrix then $\det A$ is the product of the entries on the ______ ______.

2. Answer the following questions : $2 \times 5 = 10$

(a) Compute $(1+i)^{100}$.

(b) Describe the following set explicitly and mark it on the real line

$$X = \{x \in \mathbb{R} \mid x(x-1)(x-2) < 0\}$$

(c) Consider the relation on $\mathbb{R}$ defined by $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \le y\}$. Is this relation an equivalence relation ? Justify.

(d) Find standard matrix of T, where T is a linear transformation such that $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates points about the origin through $3\pi/2$ in a counter-clockwise manner.

(e) A is 3×3 matrix with three pivot positions. Explain the following—

(i) Does $A\vec{x} = \vec{0}$ have a nontrivial solution ?

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for all $\vec{b}$ in $\mathbb{R}^3$?

3. Answer ***any four*** questions : 5×4=20

(a) If $n|q$ then prove that any root of $z^n - 1 = 0$ is a root of $z^q - 1 = 0$. Prove that the common roots of $z^m - 1 = 0$ and $z^n - 1 = 0$ are roots of $z^d - 1 = 0$ where $d =$ g.c.d (m, n)

i.e. $U_m \cap U_n = U_d$. 1+4=5

(b) Let $X = \mathbb{R} = Y$. Let $A = \{1\}$ and $B = \mathbb{R}$. Draw the sketch of $A \times B$ as a subset of $\mathbb{R}^2$.

For $A \subseteq X$ and $B \subseteq Y$ show that there may be subsets of $X \times Y$ that are not of the form $A \times B$. 2+3=5

(c) For any sets A and B, show that the following are equivalent. 5

(i) $A \subseteq B$

(ii) $A \cup B = B$

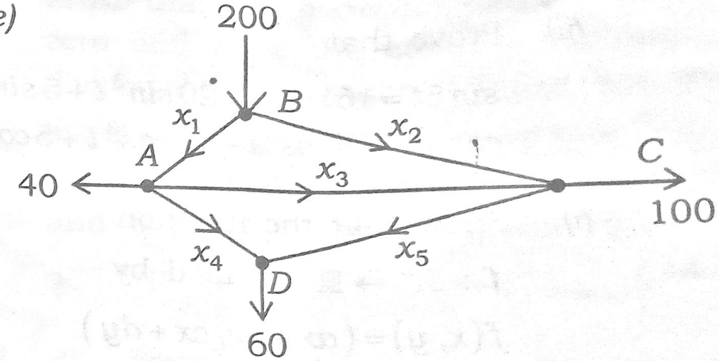
(iii) $A \cap B = A$

(iv) $B^c \subseteq A^c$

(d) Describe the solutions of the following system in parametric form. 5

$$x_1 + 3x_2 - 5x_3 = 4$$
$$x_1 + 4x_2 - 8x_3 = 7$$
$$-3x_1 - 7x_2 + 9x_3 = -6$$

(e)



For the figure above find the general traffic pattern in the freeway network (Flow rates are cars/minute)

Describe the general traffic pattern when the road whose flow is x_4 is closed. 5

(f) Use Cramer's Rule to compute the solutions of the system 5

$$2x_1 + x_2 + x_3 = 4$$
$$-x_1 + \quad + 2x_3 = 2$$
$$3x_1 + x_2 + 3x_3 = -2$$

4. Answer **any four** of the following : 10×4=40

(a) (i) Compute $z^n + \frac{1}{z^n}$ if $z + \frac{1}{z} = \sqrt{3}$. 5

(ii) Prove that 5

$$\sin 5t = 16\sin^5 t - 20\sin^3 t + 5\sin t$$

$$\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t$$

(b) (i) Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (ax + by, cx + dy)$ is a bijection of $ad - bc \neq 0$. Find the inverse of f. 5

(ii) For any sets A, B, C prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 5

(c) (i) Show that a function $f : X - Y$ is one-one if and only if $f(A \cap B) = f(A) \cap f(B)$ holds for all subsets A and B of X. 6

(ii) Give an example of a relation that is reflexive and transitive but not symmetric. 2

(iii) Produce a counter example to disapprove the statement — "For integers a, b, c if a divides bc, then a divides b or a divides c." 2

(d) Prove that if a, b are integers not both zero and d be the greatest common divisor of a and b then $\exists$ integers x, y s.t. $d = ax + by$. Further, prove that two integers m and n are relatively prime if and only if $\exists$ integers p and q s.t. $pm + qn = 1$. 6+4

(e) (i) Prove the statement using contrapositive "For integers x, y if $x + y$ is even, then x and y are both odd or both even". 2

(ii) Determine h such that the following matrix is the augmented matrix of a consistent linear system

$$\begin{bmatrix} 2 & 8 & h \\ 4 & 6 & 7 \end{bmatrix}$$ 2

Contd.

(iii) Determine if $\vec{b}$ is a linear combination of the vectors formed from the column of A. 4

$$A=\begin{bmatrix}1 & -4 & 2\\ 0 & 3 & 5\\ -2 & 8 & -4\end{bmatrix} \quad \vec{b}=\begin{bmatrix}3\\ -7\\ -3\end{bmatrix}$$

(iv) Show that the set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. 2

(f) (i) If A is a $m\times n$ matrix and $\vec{U}$, $\vec{V}$ are vectors in $\mathbb{R}^n$, C is a scalar then prove that

$A(\vec{U}+\vec{V})=A\vec{U}+A\vec{V}$

and $A(C\vec{U})=C(A\vec{U})$ 4

(ii) If A is a square $n\times n$ matrix then prove that the following statements are logically equivalent :

(a) A is an invertible matrix

(b) There is an $n\times n$ matrix C such that $CA=I$.

(c) The equation $A\vec{x}=\vec{0}$ has only the trivial solution.

(d) A has n pivot positions

(e) A is row equivalent to the $n\times n$ identity matrix. 6

(g) (i) Write the following system in matrix form and use the inverse of the co-efficient matrix to solve

$$3x_1+4x_2=3$$
$$5x_1+6x_2=7$$ 2

(ii) Let $T:\mathbb{R}^n\rightarrow\mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then show that T is invertible if and only if A is an invertible matrix. Show that the linear transformation S given by $S(\vec{x})=A^{-1}\vec{x}$ where $S:\mathbb{R}^n\rightarrow\mathbb{R}^n$, is the unique inverse of T. 4

(iii) Find the inverse of the following matrix if it exists by performing suitable row operations on the augmented matrix

$$\begin{bmatrix}1 & 0 & -2\\ -3 & 1 & 4\\ 2 & -3 & 4\end{bmatrix}$$ 4

(h) (i) Compute the determinant by co-factor expansion choosing the row or column that involves least amount of computation

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$$ 2

(ii) *State 'true'* **or** *'false' with justification :*

The $(i, j)^{th}$ cofactor of a matrix A is the matrix A_{ij} obtained from A by deleting the i^{th} row and j^{th} column of A. 2

(iii) For the matrix given below

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

(a) compute the determinant.

(b) what is determinant of an elementary row replacement of the matrix ?

(c) what is the determinant of an elementary scaling matrix with k on the diagonal ? 3

(iv) Use a determinant to decide if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, where

$$\vec{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

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